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On the laminar flow of viscous incompressible fluid through a pipe bounded by a hyperbola and a line.

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The present paper deals with laminar flow of a viscous incompressible fluid through a pipe bounded by a hyperbola and a line. The expressions, for the discharge of flux, maximum velocity and the coefficients k, k' as introduced by Boussinesqu (Bateman *et al* 1932) have been obtained. Graphs representing variations of velocity within the pipe give the final results.

Introduction

Flow through various curved pipes have been discussed by several authors Dube (1967) has discussed a problem on the hyperbolic tube and Lal (1965) has considered the flow through a tube whose cross-section is formed by the intersection of two circles.

In the present paper, flow through a pipe bounded by a hyperbola and a line, has been discussed. It is seen that if the cross sectional area is reduced, the ratio of maximum to average velocity increases slightly.

Basic Equations and Solutions

Let the axis of the pipe be along the z-axis. Assuming $u = 0 = v$ and $w = f(x, y)$, the basic equations (Lamb 1916) are

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0, \tag{1}$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{p}{\mu} \tag{2}$$

Solution of equation (2) is

$$w = \psi - \frac{p}{4\mu}(x^2+y^2), \tag{3}$$

with a boundary condition that $w = \theta$, on the boundary of the pipe and

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \qquad \ldots \quad (4)$$

Let us take $\qquad \psi = A(x^3-3xy^2)+B(x^2-y^2+x), \qquad \ldots \quad (5)$

where A and B are constants to be determined from the boundary contitions.

The equation (3) with the help of (5) becomes

$$w = A(x^3-3xy^2)+B(x^2-y^2+x)-\frac{p}{4\mu}(x^2+y^2). \qquad \dots \quad (6)$$

If one of the boundaries of the cross-sections of the pipe be $x = a$, we have $w = 0$, on $x = a$ and $(2a+1)x^2+4ax-(6a+3)y^2 = 0$.

Hence solving for the constants and substituting in (6), we get

$$w = -\frac{p}{4\mu a(2a+3)}[(x-a)\{(2a+1)x^2+4ax-(6a+3)y^2\}]. \qquad \dots \quad (7)$$

The asymptotes are given by $y = \pm 1/\sqrt{3}(x+(2a)/(2a+1))$, hence tracing the curve we get for section as shown in figure 1. The cross sectional area of the pipe is :

$$A = 2\int_0^a y\,dx = \frac{4}{\sqrt{3}}\left(\frac{4a}{2a+1}\right)^2\left[\frac{(2a+1/4)^{3/2}(2a+5/2)^{\frac{1}{2}}}{4}+\frac{(2a+1/4)^{\frac{1}{2}}(2a+5/4)^{\frac{1}{2}}}{8}\right.$$

$$\left.-\frac{1}{8}\sinh^{-1}\frac{(2a+1)^{\frac{1}{2}}}{2}\right]. \qquad \dots \quad (8)$$

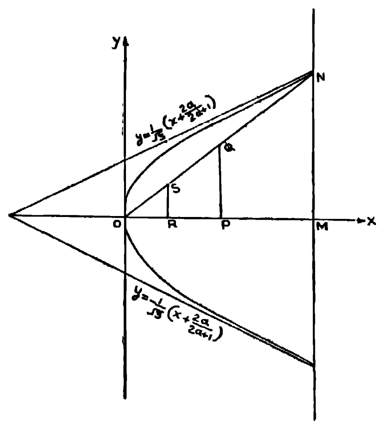

Fig. 1

The co-ordinates of the centre of gravity of the area are

$$= \frac{4a}{2a+1}\,\frac{\left[(2a+1/4)^{\frac{1}{2}}(2a+5/4)^{\frac{1}{2}}\left\{\frac{2a+1/\frac{1}{2}}{24}+\frac{(2a+1/4)^2}{6}-\frac{1}{16}\right\}+\frac{1}{16}\sinh^{-1}\frac{(2a+1)^{\frac{1}{2}}}{2}\right]}{(2a+1/4)^{\frac{1}{2}}(2a+5/4)^{\frac{1}{2}}\left\{\frac{(2a+1)}{4}+\frac{1}{8}\right\}-\frac{1}{8}\sinh^{-1}\frac{(2a+1)^{\frac{1}{2}}}{2}} \qquad (9)$$

$\bar{y} = 0.$

Now by evaluating $2\iint \rho w\,dx\,dy$ over the whole cross section, we have the discharge of flux per second as

$$
\begin{aligned}
q = {} & \rho \frac{p}{\mu} \frac{2a-3}{2a+3} \cdot \frac{1}{\sqrt{3}} \left(\frac{4a}{2a+1} \right)^4 \Bigg[\frac{(2a+5/4)^{\frac12}}{8} \frac{(2a+1)^{\frac12}}{4} \Bigg\{ \frac{(2a+1)^3}{4} + \frac{(2a+1/4)^2}{6} \\
& - \frac{5}{24} \frac{2a+1}{4} + \frac{5}{16} \Bigg\} - \frac{5}{128} \sinh^{-1} \frac{(2a+1)^{\frac12}}{2} \Bigg] - \frac{\rho p}{\mu} \frac{2a+1}{a(2a+3)3^{3/2}} \left(\frac{4a}{2a+1} \right)^5 \\
& \times \Bigg[\frac{(2a+1)^5}{4} \frac{(2a+1)^{\frac12}}{4} \frac{(2a+5)^{\frac12}}{4} \Bigg\{ \frac{(2a+1/4)^4}{10} + \frac{(2a+1/4)^3}{80} - \frac{7}{480} \frac{(2a+1)^2}{4} \\
& + \frac{7}{256} \frac{2a+1}{6} - \frac{7}{256} \Bigg\} + \frac{7}{256} \sinh^{-1} \frac{(2a+1)^{\frac12}}{2} \Bigg] + \frac{\rho p}{\mu} \frac{4a}{(2a+3)} \frac{1}{32} \left(\frac{4a}{2a+1} \right)^3 \\
& \Bigg[\frac{(2a+1)^{\frac12}}{4} \frac{(2a+5)^{\frac12}}{4} \Bigg\{ \frac{(2a+1/4)^2}{6} + \frac{2a+1/4}{24} - \frac{1}{16} \Bigg\} \frac{1}{16} \sinh^{-1} \frac{(2a+1)^{\frac12}}{2} \Bigg] \\
& + \frac{\rho p}{\mu} \frac{2a+1}{a(2a+3)3^{3/2}} \left(\frac{4a}{2a+1} \right)^5 \Bigg[\left(\frac{2a+1}{4} \right)^{\frac12} \left(\frac{2a+5}{4} \right)^{\frac12} \Bigg\{ \frac{(2a+1/4)^4}{10} + \frac{11}{80} \left(\frac{2a+1}{4} \right)^3 \\
& - \frac{2a+1/4}{128} + \frac{1}{16} \frac{(2a+1)^2}{4} + \frac{3}{256} \Bigg\} - \frac{3}{256} \sinh^{-1} \frac{(2a+1)^{\frac12}}{2} \Bigg] - \frac{\rho p}{\mu} \frac{2a+1}{(2a+3)3^{3/2}} \\
& \times \left(\frac{4a}{2a+1} \right)^4 \Bigg[\frac{(2a+1)^{\frac12}}{4} \frac{(2a+5)^{\frac12}}{4} \Bigg\{ \frac{(2a+1/4)^3}{8} + \frac{9}{48} \frac{(2a+1)^2}{4} + \frac{2a+1/4}{64} \\
& - \frac{3}{128} \Bigg\} + \frac{3}{128} \sinh^{-1} \frac{(2a+1)^{\frac12}}{2} \Bigg]. \qquad \ldots (10)
\end{aligned}
$$

From equation (7), we have for the maximum or minimum

$$\frac{\partial w}{\partial r} = \frac{-p}{4\mu a(2a+3)} [(6a+3)r^2 + (6a-4a^2)r - 4a^2] = 0 \qquad \ldots (11)$$

Or,
$$r = (4a^2-6a) \pm \{(6a-4a^2)^2 + 16a^2(6a+3)\}^{\frac12}/2(6a+3). \qquad \ldots (12)$$

Thus the maximum velocity is

$$C = \frac{-p}{4\mu a(2a+3)} [-4a^2 r + (3a-2a^2)r^2 + (2a+1)r^3], \qquad \ldots (13)$$

where r is given by (12).

The velocity at centre of gravity, $\bar{w}$ is obtained from the equations (7) and (9) by eliminating x with the help of

$$\bar{x} = x = t(\text{say}).$$

It is clear that the elimination is much more complicated.
Hence for $a = \frac12, 1, 2,$ we get

$$\bar{w} = \frac{p}{\mu} [(0.0105), (0.0333), (0.0949)]. \qquad \ldots (14)$$

From the above equation, we see that the velocity is increased with the increment of a and the ratio among $\bar{w}$ is 1:3:9 if the ratio in a is 1:2:4. Hence the ratio of velocity at centre of gravity is greater than the ratio of the span of the pipe.

Example

For the purpose of comparison with previous results we consider a numerical example. We put $a = 1$, and calculate the coefficients k, k'.

Using the notations of Boussinesq, we have

$$Q = UA = k\,\frac{p}{\mu}\,A^2, \qquad \ldots \quad (15)$$

$$C = K'A\,\frac{p,}{\mu} \qquad \ldots \quad (16)$$

where

Q = total flow per second in the pipe,

c = maximum or axial velocity in the pipe,

U = the mean velocity in the pipe,

A = cross sectional area of the pipe and

k and k' are the coefficients.

It will be easy to draw conclusions from the following table :

Table 1

	Cross sectional area	Max. vel	Flux
Circular pipe Bateman *et al* (1932)	$3.1416c^2$	$\frac{p}{4\mu}\,c^2$	$\frac{\pi c^4}{8}\,\rho\,\frac{p}{\mu}$
Elliptical pipe Lamb (1916)	$\pi a_1^2(1-e^2)^{\frac{1}{2}}$	$\frac{p}{2\mu}\,\frac{(a_1b_1)^2}{a_1^2+b_1^2}$	$\frac{\pi a_1^3 b_1^3}{a_1^2+b_1^2}\,\rho\,\frac{p}{4\mu}$
Tube bounded by two circles Lal (1965)	$1.9122b^2$	$\frac{p}{\mu}\,(0.1047)b^2$	$0.0946b^4\rho\,\frac{p}{\mu}$
Channel bounded by two hyperbolas Tripathi (1973)	$1.744d^2$	$\frac{p}{\mu}\,(0.0125)d^2$	$0.0853d^4\rho\,\frac{p}{\mu}$
Pipe bounded by a hyperbola and a line (present work)	1.05250	$0.06998\,\frac{p}{\mu}$	$0.034\,\rho\,\frac{p}{\mu}$

Thus, for the pipe under study

$$k = 0.0306, \quad k' = 0.0665, \tag{17}$$

which give the ratio of maximum and average velocity as

$$\overline{U} \quad k' \quad : 2.1732. \tag{18}$$

From the table, it can be easily seen that

$A_1 = A$ when $c^2 = 0.3352$,

$A_2 = A$ provided that $a_1^2 = 0.3352(1-e^2)^{-\frac{1}{2}}$,

$A_3 = A$ if $b^2 = 0.5504$ and

$A_4 = A$ when $d^2 = 0.6034$,

where A, A_1, A_2, A_3 and A_4 are the cross-sectional areas of the present pipe, circular pipe, elliptical pipe, tube bounded by two circles and the channel bounded by two hyperbolas, respectively.

Discussion and Physical Interpretation

The expression for velocity distribution in the pipe under study, when $a = 1$, is :

$$w(x, y) = -\frac{p}{20\mu}(x-1)(3x^2+4x-9y^2). \tag{19}$$

Let w_1 and w_2 denote velocities along x-axis and along $y = 0.88x$, which is the intersection of the line $x = 1$ and the hyperbola and is denoted as ON in figure 1. Obviously $w = 0$ at $x = 1$ and at $3x^2+4x-9y^2 = 0$, which gives $y = 0.88x$.

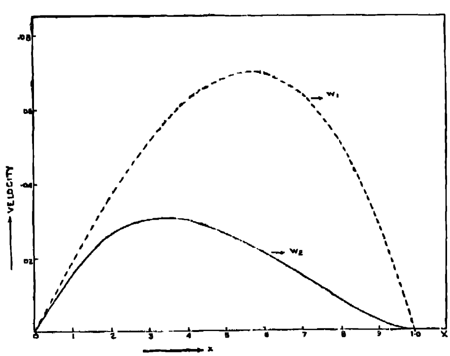

Fig. 2

The functions $w_1 \equiv w_1(x, 0)$ and $w_2 \equiv w_2(x, 0.88x)$ are plotted in figure 2. w_1 has a maximum at 0.56 and w_2 at 0.35. Thus there are two points along x at which fluid particles will have some velocity, since any line parallel to x-axis will cut the curve at two points

$$\text{Difference,} \qquad |w_1 - w_2| \equiv \frac{6.6996p}{20\mu}\,(x^{3\theta}x^{3}), \quad \text{is maximum at } x = 0.666.$$

Again, lot w_3 and w_4 denote velocities along $x = 0\ 56$ and along $x = 0.178$. The functions $w_3 \equiv w_3(0.56, y)$ and $w_4 \equiv w_4(0.178, y)$ are plotted in figuro 3. Since PQ and RS are parallel lines, w_3 is never equal to w_4. Both the velocities attain maximum at 0.

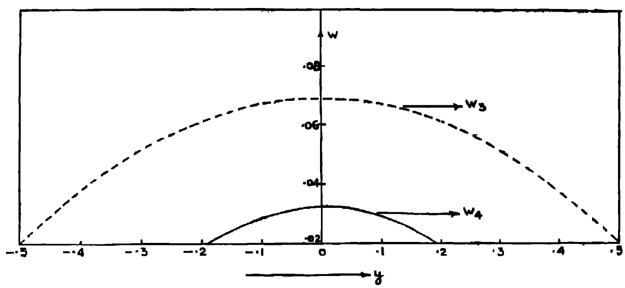

FIG. 3

$$\text{Difference,} \quad |w_3 - w_4| \equiv 3.4299y^2 + 0.7332, \quad \text{is maximum at } y = -0.459.$$

The velocity at P is $w_{3(a)} = 0.06998(p/\mu)$

and the velocity at Q is $\qquad w_{3(b)} = 0.0224\,\dfrac{p}{\mu}.$ ... (20)

Further, the velocities at R and S are :

$$w_{4(a)} = 0.0333\,\frac{p}{\mu}, \quad w_{4(b)} = 0.0238\,\frac{p}{\mu}. \qquad \text{... (21)}$$

Thus, we have

$$\begin{aligned} w_{3(b)} : w_{3(a)} &= 1{:}3.12 \\ w_{4(b)} : w_{4(a)} &= 1{:}1.40 \\ w_{3(b)} : w_{4(b)} &= 1{:}1.04 \\ w_{4(a)} : w_{3(a)} &= 1{:}2.1 \end{aligned} \qquad \text{... (22)}$$

Hence the velocity at P is three times greater than the velocity Q and the velocity at S is slightly less than the velocity at R. By this we mean that the particles near P and R are moving faster than particles near Q and S. That is, the fluid particles along x- axis are moving more speedily than particles farther from the x-axis. It is also concluded that the velocity at S is slightly greater than velocity at Q and the velocity at P is about two times larger than velocity at R.

Thus the fluid particles near P are moving with larger speed than particles near Q, R and S. There will be points given by $y = \pm 0.6$ and ± 0.3, where w_3 and w_4 are zero.

In figure 2 the variations of velocity along x-axis $\left(w_1\frac{\mu}{p}\right)$ and along the line $y = 0.88x$ $\left(w_2\frac{\mu}{p}\right)$ with difference values of x have been shown. Since $dw_1/dx = 0$ at $x = 0.56$ and $x = -0.79$ it can be easily seen that as x moves from 0 to 0.56, w_1 increases and attains maximum at $x = 0.56$ and then it decreases as x increases. Also as x moves in the negative direction of the x-axis, w_1 decreases and attains minimum *at* $x = -0.79$ and then it increases as x further decreases. Since any line parallel to x axis must cut the curve at two points, the velocities at these points are the same.

From the curve $w_2(\mu/p)$ in figure 2 we see that the velocity is maximum at $x = 0.345$ and minimum at $x = 1.041$. Thus it is concluded that the particles between $x = 0$ and $x = 0.345$ are moving faster than particles onwards to $x = 0.345$.

From the figure 2 we see that

$$\left.\begin{aligned} (w_2)_{0\cdot1} : (w_1)_{0\cdot1} &= 1 : 1.18 \\ (w_2)_{0\cdot4} : (w_1)_{0\cdot4} &= 1:1.40 \\ (w_2)_{0\cdot6} : (w_1)_{0\cdot6} &= 1:3.26 \\ (w_2)_{0\cdot9} : (w_1)_{0\cdot9} &= 1:5.29 \end{aligned}\right\} \tag{23}$$

Figure 3 gives the velocity distribution along the line through the centre of gravity $w_3(\mu/p)$ and the line through maximum point $w_4(\mu/p)$. From figure 3, it is clear that the symmetrical points have the same velocity and

$$\left.\begin{aligned} (w_4)_{0\cdot05} : (w_3)_{0\cdot05} &= 1:2.14 \\ (w_4)_{0\cdot1} : (w_3)_{0\cdot1} &= 1:2.33 \\ (w_4)_{0\cdot15} : (w_3)_{0\cdot15} &= 1:2.62 \end{aligned}\right\} \quad \ldots \quad (24)$$

From equations (23) and (24), we conclude that the ratio of the velocity at centre of gravity and the maximum velocity is nearly two for all values of y. If $y = 0$ the same ratio is obtained for the points along the lines parallel to y-axis passing through the centre of gravity and the point where the maximum velocity lies.

Conclusion

We see that the maximum velocity in the present pipe is less than the circular and elliptic pipes and the channel bounded by two hyperbolas, whereas it is more in case of a tube bounded by two circles, when the cross-sectional areas of all the pipes are equal. The maximum velocity in the present pipe is twice the velocity at centre of gravity.

Also, it is found that the discharge from the present pipe is less than the discharge from the circular and elliptic pipes, and it is more than the discharge through the sections which are formed by the intersections of (i) two circles and (ii) two hyperbolas.

It is seen that if the cross sectional area is reduced the ratio of maximum to average velocity increases. Thus it is concluded that the present pipe acts more efficiently than a tube bounded by two circles and a channel bounded by two hyperbolas.

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